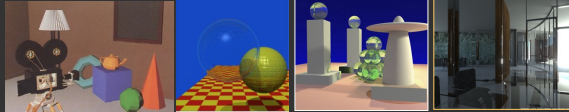


Computer Graphics II: Rendering

CSE 168 [Spr 25], Lecture 4: Rendering Equation
Ravi Ramamoorthi

<http://viscomp.ucsd.edu/classes/cse168/sp25>



1

To Do

- Homework 1 (ray tracer) due in a few days
- Next assignment direct lighting (on edX edge). Will cover that material next week

2

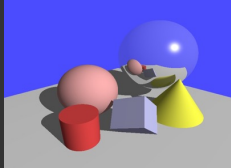
Illumination Models

Local Illumination

- Light directly from light sources to surface
- No shadows (cast shadows are a global effect)

Global Illumination: multiple bounces (indirect light)

- Hard and soft shadows
- Reflections/refractions (already seen in ray tracing)
- Diffuse and glossy interreflections (radiosity, caustics)



Some images courtesy Henrik Wann Jensen

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Caustics

Caustics: Focusing through specular surface



- Major research effort in 80s, 90s till today

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Overview of lecture

- **Theory** for all global illumination methods (ray tracing, *path tracing*, radiosity)
- We derive **Rendering Equation** [Kajiya 86]
 - Major theoretical development in field
 - Unifying framework for all global illumination
 - Introduced Path Tracing: core rendering method
- Discuss existing approaches as special cases

Fairly theoretical lecture (but important). Not well covered in textbooks (though see Eric Veach's thesis). See reading if you are interested.

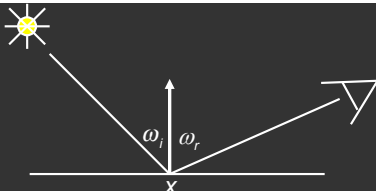
5

Outline

- **Reflectance Equation**
- **Global Illumination**
- **Rendering Equation**
- As a general Integral Equation and Operator
- Approximations (Ray Tracing, Radiosity)
- Surface Parameterization (Standard Form)

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Reflection Equation

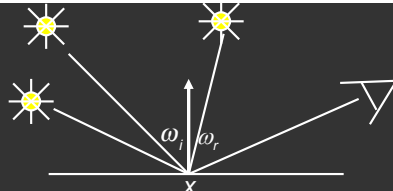


$$L_r(x, \omega_r) = L_e(x, \omega_r) + L_i(x, \omega_i) f(x, \omega_i, \omega_r) (\omega_i \cdot n)$$

Reflected Light (Output Image)	Emission	Incident Light (from light source)	BRDF	Cosine of Incident angle

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Reflection Equation



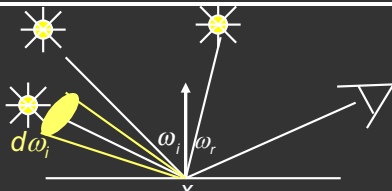
Sum over all light sources

$$L_r(x, \omega_r) = L_e(x, \omega_r) + \sum L_i(x, \omega_i) f(x, \omega_i, \omega_r) (\omega_i \cdot n)$$

Reflected Light (Output Image)	Emission	Incident Light (from light source)	BRDF	Cosine of Incident angle

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Reflection Equation



Replace sum with integral

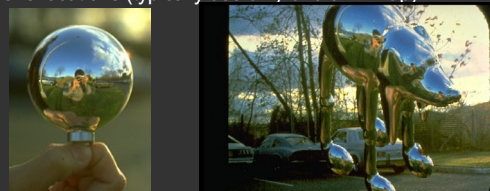
$$L_r(x, \omega_r) = L_e(x, \omega_r) + \int_{\Omega} L_i(x, \omega_i) f(x, \omega_i, \omega_r) \cos \theta_i d\omega_i$$

Reflected Light (Output Image)	Emission	Incident Light (from light source)	BRDF	Cosine of Incident angle

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Environment Maps

- Light as a function of direction, from entire environment
- Captured by photographing a chrome steel or mirror sphere
- Accurate only for one point, but distant lighting same at other scene locations (typically use only one env. map)



Blinn and Newell 1976, Miller and Hoffman, 1984
Later, Greene 86, Cabral et al. 87

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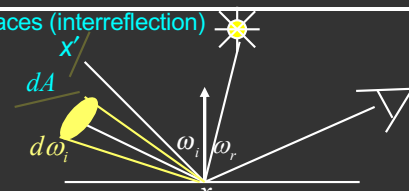
The Challenge

$$L_r(x, \omega_r) = L_e(x, \omega_r) + \int_{\Omega} L_i(x, \omega_i) f(x, \omega_i, \omega_r) \cos \theta_i d\omega_i$$

- Computing reflectance equation requires knowing the incoming radiance from surfaces
- But determining incoming radiance requires knowing the reflected radiance from surfaces

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Rendering Equation



$$L_r(x, \omega_r) = L_e(x, \omega_r) + \int_{\Omega} L_r(x', -\omega_i) f(x, \omega_i, \omega_r) \cos \theta_i d\omega_i$$

Reflected Light (Output Image)	Emission	Reflected Light	BRDF	Cosine of Incident angle
UNKNOWN	KNOWN	UNKNOWN	KNOWN	KNOWN

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Outline

- Reflectance Equation (review)
- Global Illumination
- Rendering Equation
- *As a general Integral Equation and Operator*
- *Approximations (Ray Tracing, Radiosity)*
- Surface Parameterization (Standard Form)

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Rendering Equation (Kajiya 86)

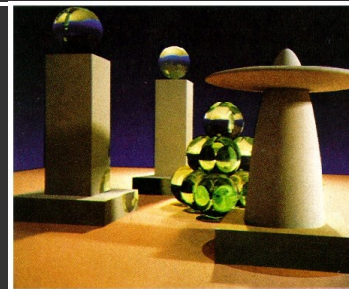


Figure 6. A sample image. All objects are neutral grey. Color on the objects is due to caustics from the green glass balls and color bleeding from the base polygon.

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Rendering Equation as Integral Equation

$$L_r(x, \omega_r) = L_e(x, \omega_r) + \int_{\Omega} L_r(x', -\omega_i) f(x, \omega_i, \omega_r) \cos \theta_i d\omega_i$$

Reflected Light (Output Image)	Emission	Reflected Light	BRDF	Cosine of Incident angle
UNKNOWN	KNOWN	UNKNOWN	KNOWN	KNOWN

Is a Fredholm Integral Equation of second kind [extensively studied numerically] with canonical form

$$l(u) = e(u) + \int l(v) K(u, v) dv$$

Kernel of equation

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Linear Operator Theory

- Linear operators act on functions like matrices act on vectors or discrete representations

$$h(u) = (M \circ f)(u)$$

M is a linear operator.
f and h are functions of u

- Basic linearity relations hold a and b are scalars
f and g are functions

$$M \circ (af + bg) = a(M \circ f) + b(M \circ g)$$

- Examples include integration and differentiation

$$(K \circ f)(u) = \int k(u, v) f(v) dv$$

$$(D \circ f)(u) = \frac{\partial f}{\partial u}(u)$$

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Linear Operator Equation

$$l(u) = e(u) + \int l(v) K(u, v) dv$$

Kernel of equation
Light Transport Operator

$$L = E + KL$$

Can be discretized to a simple matrix equation [or system of simultaneous linear equations]
(L, E are vectors, K is the light transport matrix)

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Solving the Rendering Equation

- Too hard for analytic solution, numerical methods
- Approximations, that compute different terms, accuracies of the rendering equation
- Two basic approaches are ray tracing, radiosity. More formally, Monte Carlo and Finite Element. Today Monte Carlo path tracing is core rendering method
- Monte Carlo techniques sample light paths, form statistical estimate (example, path tracing)
- Finite Element methods discretize to matrix equation

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Solving the Rendering Equation

- General linear operator solution. Within raytracing:
- General class numerical *Monte Carlo* methods
- Approximate set of all paths of light in scene

$$L = E + KL$$

$$IL - KL = E$$

$$(I - K)L = E$$

$$L = (I - K)^{-1}E$$

Binomial Theorem

$$L = (I + K + K^2 + K^3 + \dots)E$$

$$L = E + KE + K^2E + K^3E + \dots$$

Term n corresponds to n bounces of light

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Ray Tracing

$$L = E + KE + K^2E + K^3E + \dots$$

Emission directly
From light sources

Direct Illumination
on surfaces

Global Illumination
(One bounce indirect)
[Mirrors, Refraction]

(Two bounce indirect)
[Caustics etc]

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Ray Tracing

$$L = E + KE + K^2E + K^3E + \dots$$

Emission directly
From light sources

Direct Illumination
on surfaces

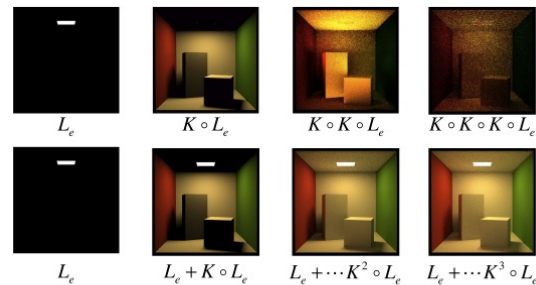
OpenGL Shading

Global Illumination
(One bounce indirect)
[Mirrors, Refraction]

(Two bounce indirect)
[Caustics etc]

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Successive Approximation



CS348B Lecture 13

Pat Hanrahan, Spring 2009

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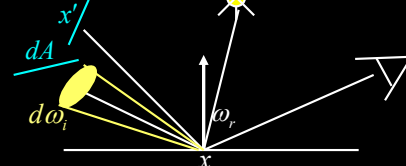
Outline

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Rendering Equation

Surfaces (interreflection)



$$\omega_i \sim x' - x$$

$$L_r(x, \omega_r) = L_e(x, \omega_r) + \int_{\Omega} L_r(x', -\omega_i) f(x, \omega_i, \omega_r) \cos \theta_i d\omega_i$$

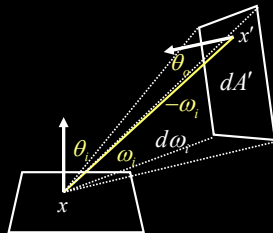
Reflected Light (Output Image)	Emission	Reflected Light	BRDF	Cosine of Incident angle
UNKNOWN	KNOWN	UNKNOWN	KNOWN	KNOWN

24

Change of Variables

$$L_r(x, \omega_r) = L_e(x, \omega_r) + \int_{\Omega} L_r(x', -\omega_i) f(x, \omega_i, \omega_r) \cos \theta_i d\omega_i$$

Integral over angles sometimes insufficient. Write integral in terms of surface radiance only (change of variables)



$$d\omega_i = \frac{dA' \cos \theta_o}{|x - x'|^2}$$

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Change of Variables

$$L_r(x, \omega_r) = L_e(x, \omega_r) + \int_{\Omega} L_r(x', -\omega_i) f(x, \omega_i, \omega_r) \cos \theta_i d\omega_i$$

Integral over angles sometimes insufficient. Write integral in terms of surface radiance only (change of variables)

$$L_r(x, \omega_r) = L_e(x, \omega_r) + \int_{\text{all } x' \text{ visible to } x} L_r(x', -\omega_i) f(x, \omega_i, \omega_r) \frac{\cos \theta_i \cos \theta_o}{|x - x'|^2} dA'$$

$$d\omega_i = \frac{dA' \cos \theta_o}{|x - x'|^2}$$

$$G(x, x') = G(x', x) = \frac{\cos \theta_i \cos \theta_o}{|x - x'|^2}$$

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Rendering Equation: Standard Form

$$L_r(x, \omega_r) = L_e(x, \omega_r) + \int_{\Omega} L_r(x', -\omega_i) f(x, \omega_i, \omega_r) \cos \theta_i d\omega_i$$

Integral over angles sometimes insufficient. Write integral in terms of surface radiance only (change of variables)

$$L_r(x, \omega_r) = L_e(x, \omega_r) + \int_{\text{all } x' \text{ visible to } x} L_r(x', -\omega_i) f(x, \omega_i, \omega_r) \frac{\cos \theta_i \cos \theta_o}{|x - x'|^2} dA'$$

Domain integral awkward. Introduce binary visibility fn V

$$L_r(x, \omega_r) = L_e(x, \omega_r) + \int_{\text{all surfaces } x'} L_r(x', -\omega_i) f(x, \omega_i, \omega_r) G(x, x') V(x, x') dA'$$

Same as equation 2.52 Cohen Wallace. It swaps primed and unprimed, omits angular args of BRDF, - sign.

Same as equation above 19.3 in Shirley, except he has no emission, slightly diff. notation

$$d\omega_i = \frac{dA' \cos \theta_o}{|x - x'|^2}$$

$$G(x, x') = G(x', x) = \frac{\cos \theta_i \cos \theta_o}{|x - x'|^2}$$

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Radiosity Equation

$$L_r(x, \omega_r) = L_e(x, \omega_r) + \int_{\text{all surfaces } x'} L_r(x', -\omega_i) f(x, \omega_i, \omega_r) G(x, x') V(x, x') dA'$$

Drop angular dependence (diffuse Lambertian surfaces)

$$L_r(x) = L_e(x) + f(x) \int_{\Omega} L_r(x') G(x, x') V(x, x') dA'$$

Change variables to radiosity (B) and albedo (rho)

$$B(x) = E(x) + \rho(x) \int_{\Omega} B(x') \frac{G(x, x') V(x, x')}{\pi} dA'$$

Expresses conservation of light energy at all points in space

Same as equation 2.54 in Cohen Wallace handout (read sec 2.6.3) Ignore factors of pi which can be absorbed.

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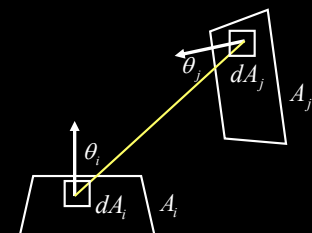
Discretization and Form Factors

$$B(x) = E(x) + \rho(x) \int_{\Omega} B(x') \frac{G(x, x') V(x, x')}{\pi} dA'$$

$$B_i = E_i + \rho_i \sum_j B_j F_{j \rightarrow i} \frac{A_j}{A_i}$$

F is the **form factor**. It is dimensionless and is the fraction of energy leaving the entirety of patch j (multiply by area of j to get total energy) that arrives anywhere in the entirety of patch i (divide by area of i to get energy per unit area or radiosity).

Form Factors



$$A_i F_{i \rightarrow j} = A_j F_{j \rightarrow i} = \iint \frac{G(x, x') V(x, x')}{\pi} dA_i dA_j$$

$$G(x, x') = G(x', x) = \frac{\cos \theta_i \cos \theta_o}{|x - x'|^2}$$

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Matrix Equation

$$B_i = E_i + \rho_i \sum_j B_j F_{j \rightarrow i} \frac{A_j}{A_i}$$

$$A_i F_{i \rightarrow j} = A_j F_{j \rightarrow i} = \iint \frac{G(x, x') V(x, x')}{\pi} dA_i dA_j$$

$$B_i = E_i + \rho_i \sum_j B_j F_{i \rightarrow j}$$

$$B_i - \rho_i \sum_j B_j F_{i \rightarrow j} = E_i$$

$$\sum_j M_{ij} B_j = E_i \quad MB = E \quad M_{ij} = I_{ij} - \rho_i F_{i \rightarrow j}$$

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Radiosity



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Radiosity Epitaph

- Very hot topic (about 1985-1994). Some of the most beautiful images, greatest researchers; at one point 50% of SIGGRAPH papers
- But visibility, meshing, discontinuities, complex BRDFs, volumes were all difficult problems
- Since mid-90s, Monte Carlo (rather than finite element) methods were preferred (going back to Monte Carlo Path Tracing in Kajiya86)
- Today, path tracing entirely method of choice, widely used in industry. This is what 168 teaches

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Summary

- **Theory** for all global illumination methods (ray tracing, path tracing, radiosity)
- We derive **Rendering Equation** [Kajiya 86]
 - Major theoretical development in field
 - Unifying framework for all global illumination
- Discuss existing approaches as special cases
- *Rest of Course: Solving the Rendering Equation (numerically using Monte Carlo Path Tracing)*

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