Computer Graphics II: Rendering

CSE 168[Spr 25],Lecture 11: Fourier Analysis, Sampling
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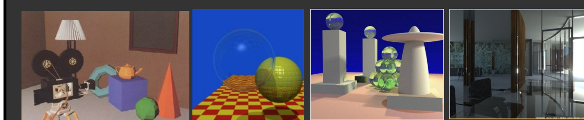
http://viscomp.ucsd.edu/classes/cse168/sp25



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To Do

- Start immediately on homework 4.
- Start thinking about final project
- This lecture gives core background on sampling and signal-processing (bear in mind image processing)

Some slides courtesy Pat Hanrahan

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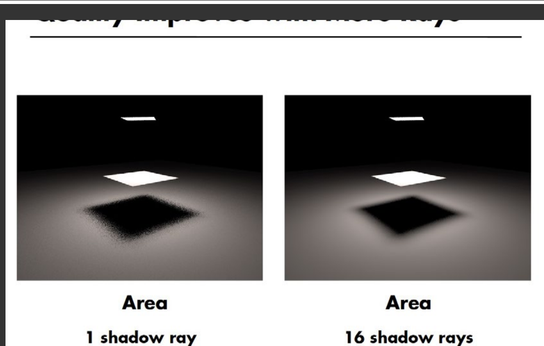
Quality Improves with More Rays



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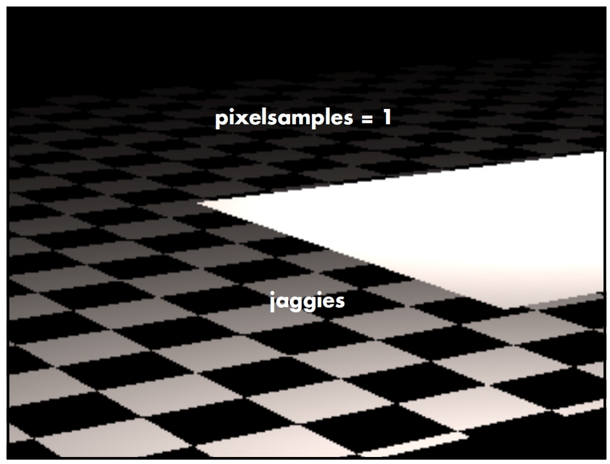

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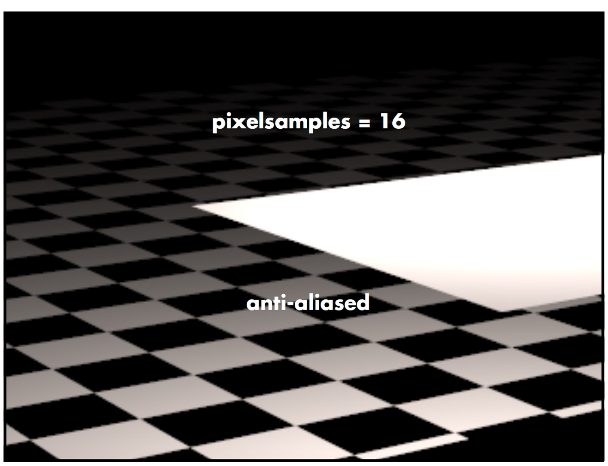

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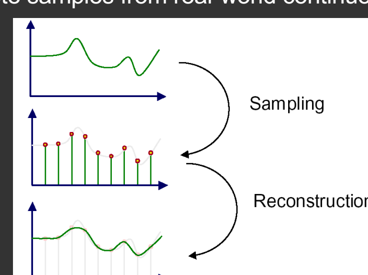
Sampling and Reconstruction

- An image is a 2D array of samples
- Discrete samples from real-world continuous signal



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Sampling and Reconstruction

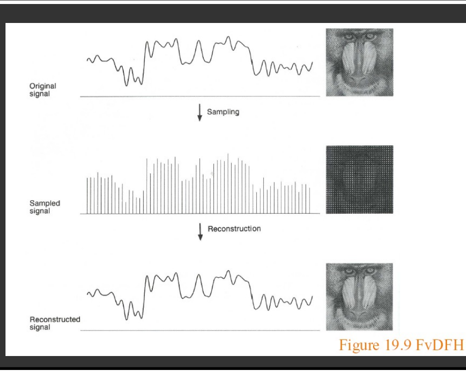

Figure 19.9 FvDFH

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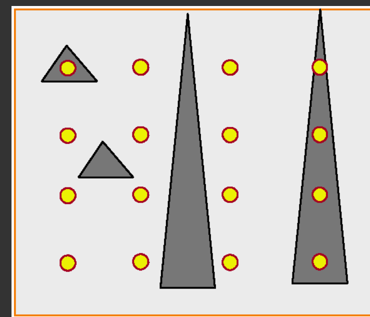
(Spatial) Aliasing



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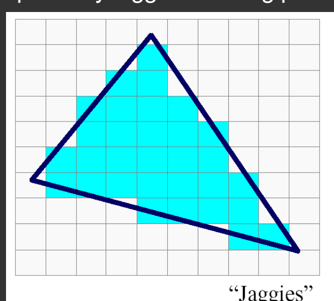
(Spatial) Aliasing

- Jaggies probably biggest aliasing problem



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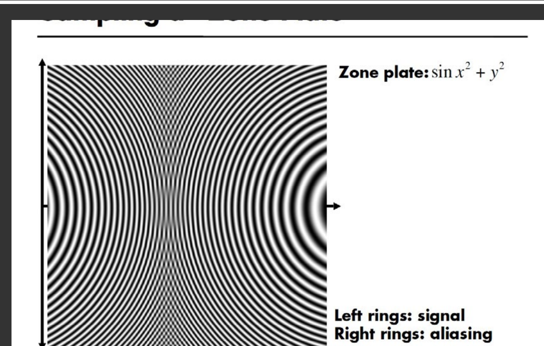
Sampling a Zone Plate



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Sampling and Aliasing

- Artifacts due to undersampling or poor reconstruction
- Formally, high frequencies masquerading as low
- E.g. high frequency line as low freq jaggies

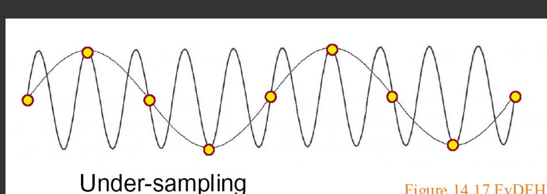

Figure 14.17 FvDFH

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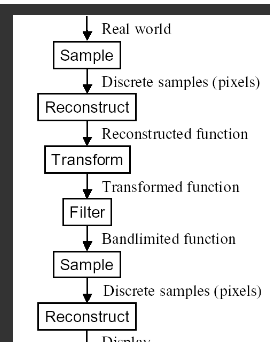
Image Processing pipeline



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Motivation

- Formal analysis of sampling and reconstruction
- Important theory (signal-processing) for graphics
- Also relevant in rendering, modeling, animation

- Note: Fourier Analysis useful for understanding, but image processing often done in spatial domain

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Ideas

- Signal (function of time generally, here of space)
- Continuous: defined at all points; discrete: on a grid
- High frequency: rapid variation; Low Freq: slow variation
- Images are converting continuous to discrete. Do this sampling as best as possible.
- Signal processing theory tells us how best to do this
- Based on concept of frequency domain Fourier analysis

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Sampling Theory

Analysis in the frequency (not spatial) domain
- Sum of sine waves, with possibly different offsets (phase)
- Each wave different frequency, amplitude

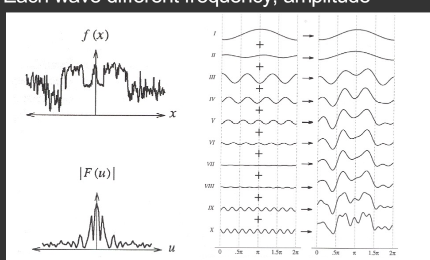

Figure 2.6 Wolberg

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Fourier Transform

- Tool for converting from spatial to frequency domain

$$f(x) = \sum_{u=-\infty}^{+\infty} F(u)e^{2\pi iux}$$

$$e^{2\pi iux} = \cos(2\pi ux) + i\sin(2\pi ux)$$

- Or vice versa $\quad i = \sqrt{-1}$
- One of most important mathematical ideas
- Computational algorithm: Fast Fourier Transform
  - One of 10 great algorithms scientific computing
  - Makes Fourier processing possible (images etc.)
  - Not discussed here, but look up if interested

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Fourier Transform

- Simple case, function sum of sines, cosines

$$f(x) = \sum_{u=-\infty}^{+\infty} F(u)e^{2\pi iux}$$

$$F(u) = \int_0^1 f(x)e^{-2\pi iux}dx$$

- Continuous infinite case

Forward Transform: $$F(u) = \int_{-\infty}^{\infty} f(x)e^{-2\pi iux}dx$$

Inverse Transform: $$f(x) = \int_{-\infty}^{+\infty} F(u)e^{2\pi iux}du$$

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Fourier Transform

- Simple case, function sum of sines, cosines

$$f(x) = \sum_{u=-\infty}^{+\infty} F(u)e^{2\pi iux}$$

$$F(u) = \int_0^1 f(x)e^{-2\pi iux}dx$$

- Discrete case

$$F(u) = \sum_{x=0}^{x=N-1} f(x)\left[\cos(2\pi ux/N) - i\sin(2\pi ux/N)\right], \quad 0 \le u \le N-1$$

$$f(x) = \frac{1}{N}\sum_{u=0}^{u=N-1} F(u)\left[\cos(2\pi ux/N) + i\sin(2\pi ux/N)\right], \quad 0 \le x \le N-1$$

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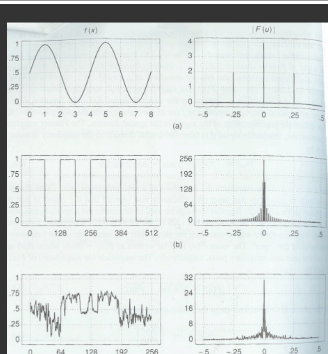
Fourier Transform: Examples 1

Single sine curve
(+constant DC term)



$$f(x) = \sum_{u=-\infty}^{+\infty} F(u) e^{2\pi iux}$$

$$F(u) = \int_0^1 f(x) e^{-2\pi iux} dx$$

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Fourier Transform Examples 2

Forward Transform: $F(u) = \int_{-\infty}^{\infty} f(x) e^{-2\pi iux} dx$

Inverse Transform: $f(x) = \int_{-\infty}^{+\infty} F(u) e^{2\pi iux} du$

- Common examples

$f(x)$	$F(u)$
$\delta(x - x_0)$	$e^{-2\pi iux_0}$
1	$\delta(u)$
e^{-ax^2}	$\sqrt{\pi / a}\, e^{-\pi^2 u^2 / a}$

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Fourier Transform Properties

Forward Transform: $F(u) = \int_{-\infty}^{\infty} f(x) e^{-2\pi iux} dx$

Inverse Transform: $f(x) = \int_{-\infty}^{+\infty} F(u) e^{2\pi iux} du$

- Common properties
  - Linearity: $F(af(x) + bg(x)) = aF(f(x)) + bF(g(x))$
  - Derivatives: [integrate by parts] $F(f'(x)) = \int_{-\infty}^{\infty} f'(x) e^{-2\pi iux} dx = 2\pi iuF(u)$
  - 2D Fourier Transform
    Forward Transform: $F(u,v) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x,y) e^{-2\pi iux} e^{-2\pi ivy} dxdy$
    Inverse Transform: $f(x,y) = \int_{-\infty}^{\infty} \int_{-\infty}^{+\infty} F(u,v) e^{2\pi iux} e^{2\pi ivy} dudv$
- Convolution (next)

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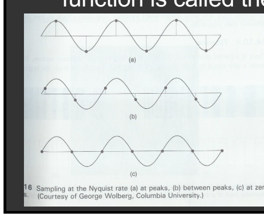
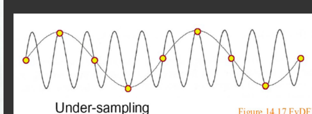
Sampling Theorem, Bandlimiting

- A signal can be reconstructed from its samples, if the original signal has no frequencies above half the sampling frequency – Shannon
- The minimum sampling rate for a bandlimited function is called the Nyquist rate





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Sampling Theorem, Bandlimiting

- A signal can be reconstructed from its samples, if the original signal has no frequencies above half the sampling frequency – Shannon
- The minimum sampling rate for a bandlimited function is called the Nyquist rate
- A signal is bandlimited if the highest frequency is bounded. This frequency is called the bandwidth
- In general, when we transform, we want to filter to bandlimit before sampling, to avoid aliasing

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Antialiasing

- Sample at higher rate
  - Not always possible
  - Real world: lines have infinitely high frequencies, can't sample at high enough resolution
- Prefilter to bandlimit signal
  - Low-pass filtering (blurring)
  - Trade blurriness for aliasing

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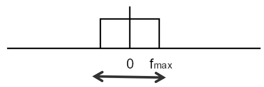
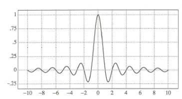
Ideal bandlimiting filter

- Formal derivation is homework exercise
  - Frequency domain



  - Spatial domain



if full width $f_{max} = 1$

$$Sinc(x) = \frac{\sin \pi x}{\pi x}$$

Figure 4.5 Wolberg

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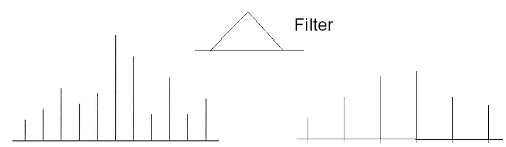
Convolution 1

- Spatial domain: output pixel is weighted sum of pixels in neighborhood of input image
  - Pattern of weights is the "filter"



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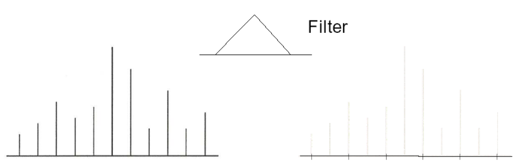
Convolution 2

- Example 1:



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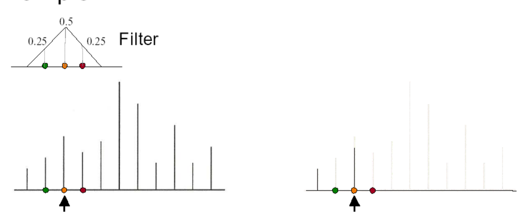
Convolution 3

- Example 1:



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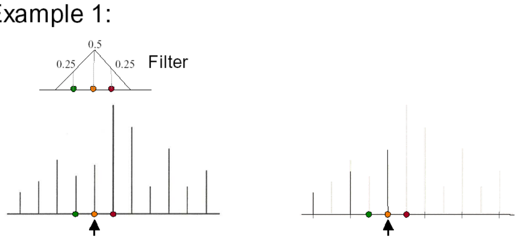
Convolution 4

- Example 1:



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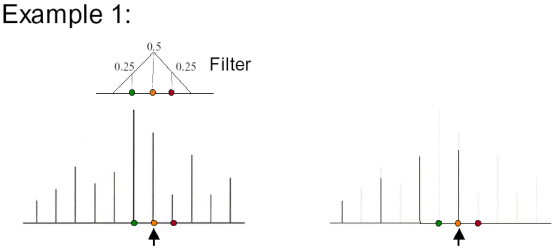
Convolution 5

- Example 1:



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Convolution in Frequency Domain

Forward Transform: $F(u)=\int_{-\infty}^{\infty} f(x)e^{-2\pi iux}dx$

Inverse Transform: $f(x)=\int_{-\infty}^{+\infty} F(u)e^{2\pi iux}du$

- Convolution (f is signal ; g is filter [or vice versa])

$$h(y)=\int_{-\infty}^{+\infty} f(x)g(y-x)dx=\int_{-\infty}^{+\infty} g(x)f(y-x)dx$$

$$h=f*g \;\; or \;\; f\otimes g$$

- Fourier analysis (frequency domain multiplication) $H(u)=F(u)G(u)$

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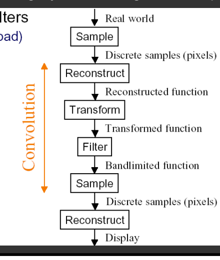
Practical Image Processing

- Discrete convolution (in spatial domain) with filters for various digital signal processing operations
- Easy to analyze, understand effects in frequency domain
  - E.g. blurring or bandlimiting by convolving with low pass filter

- Finite low-pass filters
  - Point sampling (bad)
  - Triangle filter
  - Gaussian filter



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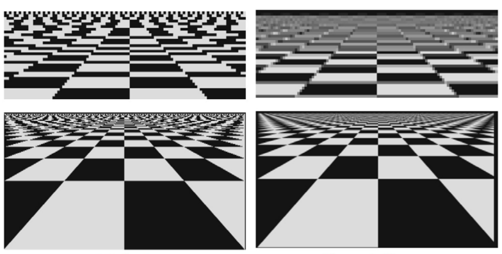
Point vs Area Sampling



Point

Exact Area

Checkerboard sequence by Tom Duff

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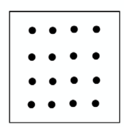
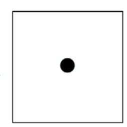
Uniform Supersampling

Increasing the number of samples moves each copy of the spectra further apart, thus there is less overlap

This reduces, but does not eliminate, aliasing



$$Pixel = \sum_s w_s \cdot Sample_s$$



Samples

Pixel

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Non-uniform Sampling

Uniform sampling

- The spectrum of uniformly spaced samples is also a set of uniformly spaced spikes
- Multiplying the signal by the sampling pattern corresponds to placing a copy of the spectrum at each spike (in freq. space)
- Aliases are coherent, and very noticable

Non-uniform sampling

- Samples at non-uniform locations have a different spectrum; a single spike plus noise
- Sampling a signal in this way converts aliases into broadband noise
- Noise is incoherent, and much less objectionable
- May cause error in the integral

CS348b Lecture 8 Pat Hanrahan / Matt Pharr, Spring 2019

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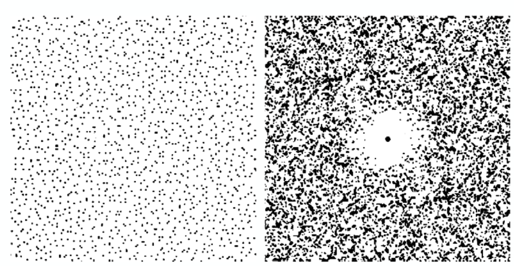
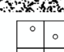
Jittered Sampling



Add uniform random jitter to each sample



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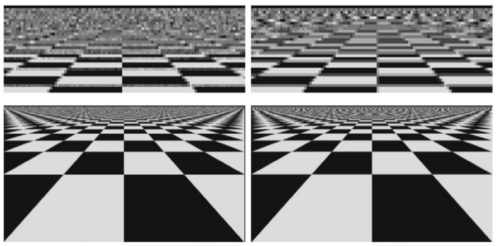
Jittered vs Uniform Supersampling



4x4 Jittered Sampling 4x4 Uniform

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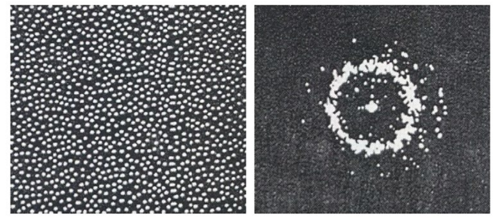
Distribution of Extrafoveal Cones



Monkey eye cone distribution Fourier transform

Yellot

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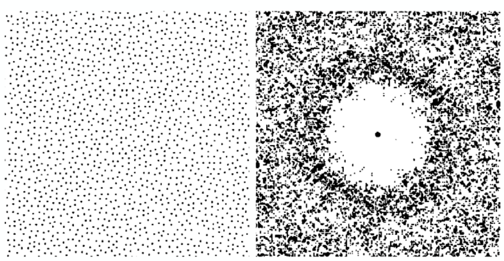
Poisson Disk Sampling



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