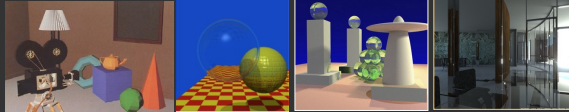


## Computer Graphics II: Rendering

CSE 168 [Spr 24], Lecture 5: Monte Carlo Integration  
Ravi Ramamoorthi

<http://viscomp.ucsd.edu/classes/cse168/sp24>



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## To Do

- Homework 2 (Direct Lighting) due Apr 25
- Assignment is on UCSD Online
- START EARLY (NOW)

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## Motivation

Rendering = integration

- Reflectance equation: Integrate over incident illumination
- Rendering equation: Integral equation

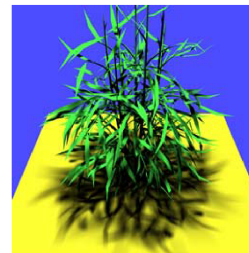
Many sophisticated shading effects involve integrals

- Antialiasing
- Soft shadows
- Indirect illumination
- Caustics

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## Example: Soft Shadows

$$E(x) = \int_{H^2} L_i(x, \omega) \cos \theta d\omega$$



**Challenges**

- **Visibility and blockers**
- **Varying light distribution**
- **Complex source geometry**

Source: Agrawala, Ramamoorthi, Heirich, Moll, 2000

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## Monte Carlo

- Algorithms based on statistical sampling and random numbers
- Coined in the beginning of 1940s. Originally used for neutron transport, nuclear simulations
  - Von Neumann, Ulam, Metropolis, ...
- Canonical example: 1D integral done numerically
  - Choose a set of random points to evaluate function, and then average (expectation or statistical average)

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## Monte Carlo Algorithms

**Advantages**

- Robust for complex integrals in computer graphics (irregular domains, shadow discontinuities and so on)
- Efficient for high dimensional integrals (common in graphics: time, light source directions, and so on)
- Quite simple to implement
- Work for general scenes, surfaces
- Easy to reason about (but care taken re statistical bias)

**Disadvantages**

- Noisy
- Slow (many samples needed for convergence)
- Not used if alternative analytic approaches exist (but those are rare)

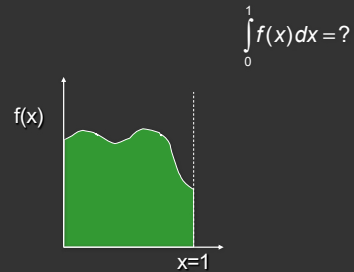
6

## Outline

- Motivation
- Overview, 1D integration
- Basic probability and sampling
- Monte Carlo estimation of integrals

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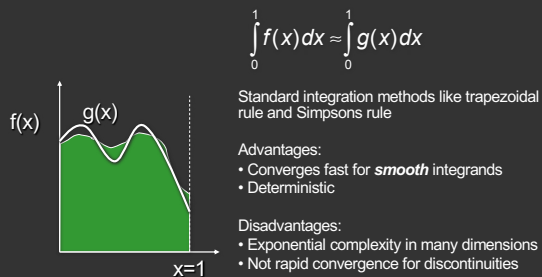
## Integration in 1D



Slide courtesy of Peter Shirley

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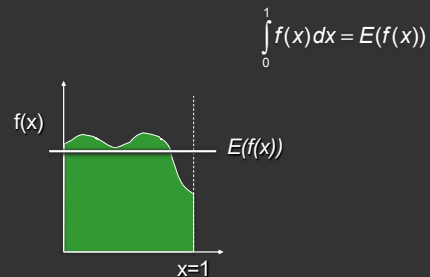
## We can approximate



Slide courtesy of Peter Shirley

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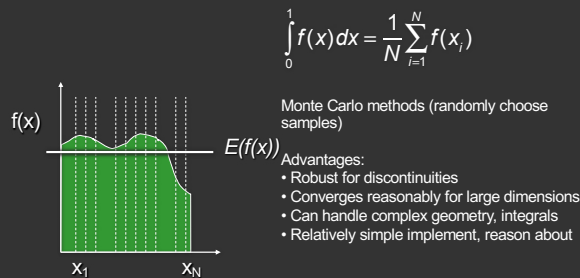
## Or we can average



Slide courtesy of Peter Shirley

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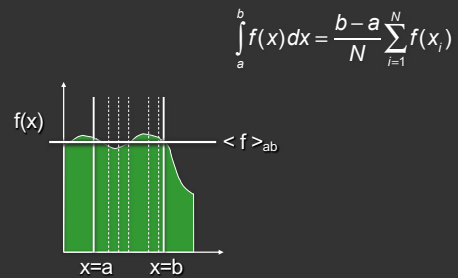
## Estimating the average



Slide courtesy of Peter Shirley

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## Other Domains



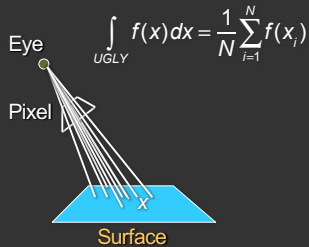
Slide courtesy of Peter Shirley

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## Multidimensional Domains

Same ideas apply for integration over ...

- Pixel areas
- Surfaces
- Projected areas
- Directions
- Camera apertures
- Time
- Paths



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## Outline

- Motivation
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## Random Variables

- Describes possible outcomes of an experiment
- In discrete case, e.g. value of a dice roll [ $x = 1-6$ ]
- Probability  $p$  associated with each  $x$  ( $1/6$  for dice)
- Continuous case is obvious extension

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## Expected Value

- Expectation Discrete:  $E(x) = \sum_{i=1}^n p_i x_i$
- Continuous:  $E(x) = \int_0^1 p(x) f(x) dx$

- For Dice example:

$$E(x) = \sum_{i=1}^n \frac{1}{6} x_i = \frac{1}{6} (1 + 2 + 3 + 4 + 5 + 6) = 3.5$$

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## Continuous Probability Distributions

**PDF**  $p(x)$

$$p(x) \geq 0$$

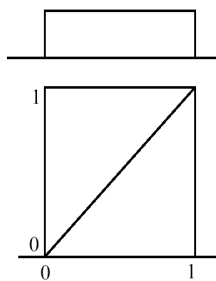
**CDF**  $P(x)$

$$P(x) = \int_0^x p(x) dx$$

$$P(x) = \Pr(X < x) \quad P(1) = 1$$

$$\Pr(\alpha \leq X \leq \beta) = \int_{\alpha}^{\beta} p(x) dx = P(\beta) - P(\alpha)$$

**Uniform**



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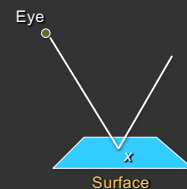
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## Sampling Techniques

Problem: how do we generate random points/directions during path tracing?

- Non-rectilinear domains
- Importance (BRDF)
- Stratified

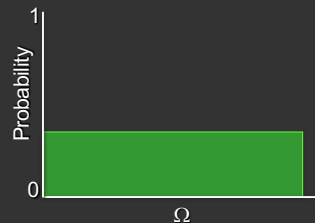


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## Generating Random Points

Uniform distribution:

- Use random number generator

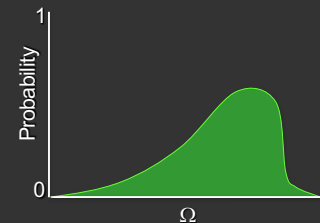


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## Generating Random Points

Specific probability distribution:

- Function inversion
- Rejection
- Metropolis



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## Common Operations

Want to **sample** probability distributions

- Draw samples distributed according to probability
- Useful for integration, picking important regions, etc.

Common distributions

- Disk or circle
- Uniform
- Upper hemisphere for visibility
- Area luminaire
- Complex lighting like an environment map
- Complex reflectance like a BRDF

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## Sampling Continuous Distributions

Cumulative probability distribution function

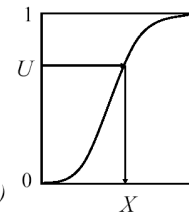
$$P(x) = \Pr(X < x)$$

Construction of samples

$$\text{Solve for } X = P^{-1}(U)$$

Must know:

- The integral of  $p(x)$
- The inverse function  $P^{-1}(x)$

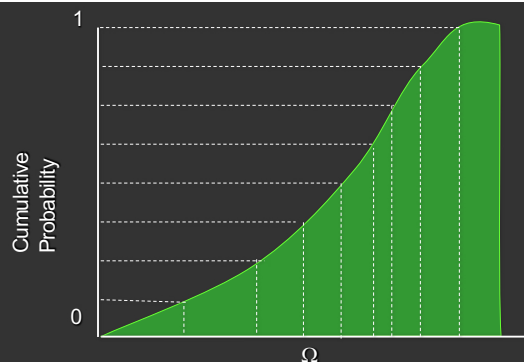


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## Generating Random Points



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## Example: Power Function

Assume

$$p(x) = (n+1)x^n$$

$$\int_0^1 x^n dx = \frac{x^{n+1}}{n+1} \Big|_0^1 = \frac{1}{n+1}$$

$$P(x) = x^{n+1}$$

$$X \sim p(x) \Rightarrow X = P^{-1}(U) = \sqrt[n+1]{U}$$

Trick

$$Y = \max(U_1, U_2, \dots, U_n, U_{n+1})$$

$$\Pr(Y < x) = \prod_{i=1}^{n+1} \Pr(U_i < x) = x^{n+1}$$

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## Sampling a Circle

$$A = \int_0^{2\pi} \int_0^1 r \, dr \, d\theta = \int_0^1 r \, dr \int_0^{2\pi} d\theta = \left( \frac{r^2}{2} \right) \Big|_0^1 \theta \Big|_0^{2\pi} = \pi$$

$$p(r, \theta) \, dr \, d\theta = \frac{1}{\pi} r \, dr \, d\theta \Rightarrow p(r, \theta) = \frac{r}{\pi}$$

$$p(r, \theta) = p(r)p(\theta)$$

$$p(\theta) = \frac{1}{2\pi}$$

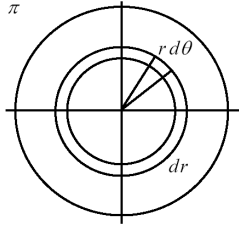
$$P(\theta) = \frac{1}{2\pi} \theta$$

$$p(r) = 2r$$

$$P(r) = r^2$$

$$\theta = 2\pi U_1$$

$$r = \sqrt{U_2}$$



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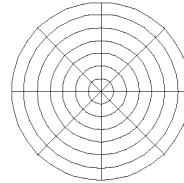
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## Sampling a Circle

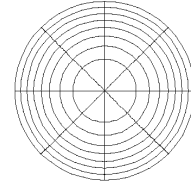
**WRONG** ≠ Equi-Areal

**RIGHT** = Equi-Areal



$$\theta = 2\pi U_1$$

$$r = U_2$$



$$\theta = 2\pi U_1$$

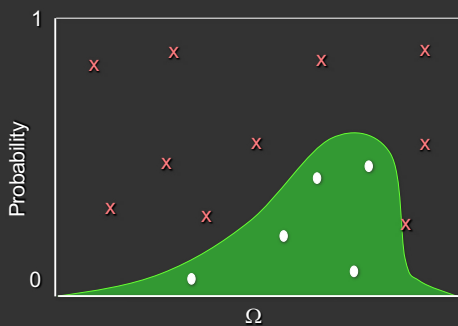
$$r = \sqrt{U_2}$$

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## Rejection Sampling

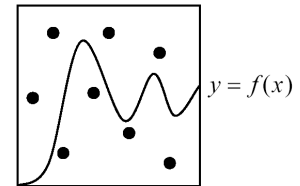


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## Rejection Methods

$$I = \int_0^1 f(x) \, dx$$

$$= \iint_{y < f(x)} dx \, dy$$



**Algorithm**

**Pick**  $U_1$  and  $U_2$

**Accept**  $U_1$  if  $U_2 < f(U_1)$

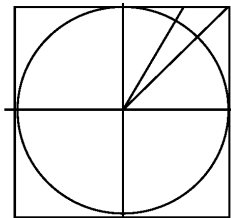
**Wasteful?** Efficiency = Area / Area of rectangle

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## Sampling a Circle: Rejection



do {

$$X = 1 - 2 * U_1$$

$$Y = 1 - 2 * U_2$$

while (  $X^2 + Y^2 > 1$  )

May be used to pick random 2D directions

Circle techniques may also be applied to the sphere

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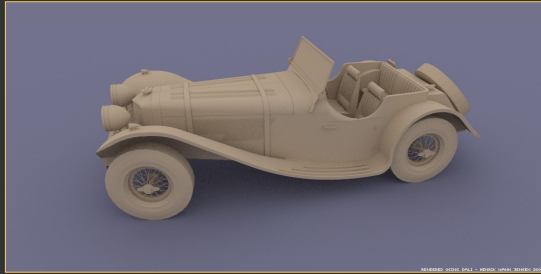
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## Outline

- Motivation
- Overview, 1D integration
- Basic probability and sampling
- Monte Carlo estimation of integrals

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## Monte Carlo Path Tracing



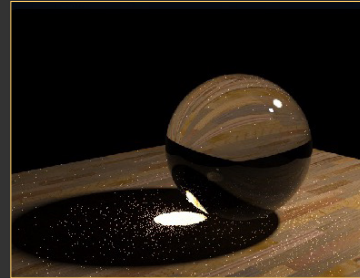
Big diffuse light source, 20 minutes

Motivation for rendering in graphics: Covered in detail in next lecture

Jensen

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## Monte Carlo Path Tracing

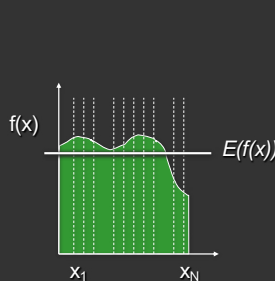


1000 paths/pixel

Jensen

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## Estimating the average



$$\int_0^1 f(x) dx = \frac{1}{N} \sum_{i=1}^N f(x_i)$$

Monte Carlo methods (randomly choose samples)

Advantages:

- Robust for discontinuities
- Converges reasonably for large dimensions
- Can handle complex geometry, integrals
- Relatively simple implement, reason about

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## Monte Carlo Integration

|                                      |                                      |
|--------------------------------------|--------------------------------------|
| <b>Definite integral</b>             | $I(f) \equiv \int_0^1 f(x) dx$       |
| <b>Expectation of <math>f</math></b> | $E[f] \equiv \int_0^1 f(x) p(x) dx$  |
| <b>Random variables</b>              | $X_i \sim p(x)$<br>$Y_i = f(X_i)$    |
| <b>Estimator</b>                     | $F_N = \frac{1}{N} \sum_{i=1}^N Y_i$ |

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## Unbiased Estimator

$$E[F_N] = I(f)$$

### Properties

$$E[\sum_i Y_i] = \sum_i E[Y_i]$$

$$E[aY] = aE[Y]$$

$$\begin{aligned} E[F_N] &= E\left[\frac{1}{N} \sum_{i=1}^N Y_i\right] \\ &= \frac{1}{N} \sum_{i=1}^N E[Y_i] = \frac{1}{N} \sum_{i=1}^N E[f(X_i)] \\ &= \frac{1}{N} \sum_{i=1}^N \int_0^1 f(x) p(x) dx \\ &= \frac{1}{N} \sum_{i=1}^N \int_0^1 f(x) dx \\ &= \int_0^1 f(x) dx \end{aligned}$$

Assume uniform probability distribution for now

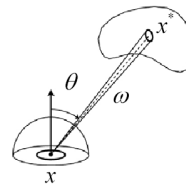
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## Direct Lighting - Directional Sampling

$$E(x) = \int_{\Omega} L(x, \omega) \cos \theta d\omega$$



Ray intersection  $x^*(x, \omega)$

Sample  $\omega$  uniformly by  $\Omega$

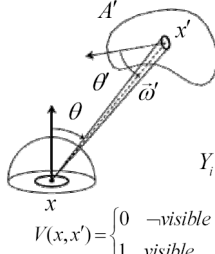
$$Y_i = L(x^*(x, \omega_i), -\omega_i) \cos \theta_i 2\pi$$

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## Direct Lighting - Area Sampling

$$E(x) = \int_{\Omega} L_i(x, \omega) \cos \theta d\omega = \int_{A'} L_o(x', \omega') V(x, x') \frac{\cos \theta \cos \theta'}{|x - x'|^2} dA'$$


**Ray direction**  $\omega' = x - x'$

**Sample  $x'$  uniformly by  $A'$**

$$Y_i = L_o(x'_i, \omega'_i) V(x, x'_i) \frac{\cos \theta \cos \theta'_i}{|x - x'_i|^2} A$$

$$V(x, x') = \begin{cases} 0 & \text{--visible} \\ 1 & \text{visible} \end{cases}$$

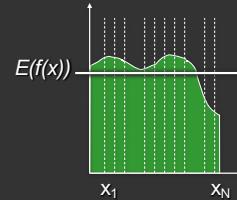
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## Variance

$$\text{Var}[f(x)] = \frac{1}{N} \sum_{i=1}^N [f(x_i) - E(f(x))]^2$$



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## Variance

### Definition

$$\begin{aligned} V[Y] &\equiv E[(Y - E[Y])^2] \\ &= E[Y^2 - 2YE[Y] + E[Y]^2] \\ &= E[Y^2] - E[Y]^2 \end{aligned}$$

### Properties

$$\begin{aligned} V[\sum_i Y_i] &= \sum_i V[Y_i] \\ V[aY] &= a^2 V[Y] \end{aligned}$$

### Variance decreases with sample size

$$V[\frac{1}{N} \sum_{i=1}^N Y_i] = \frac{1}{N^2} \sum_{i=1}^N V[Y_i] = \frac{1}{N} V[Y]$$

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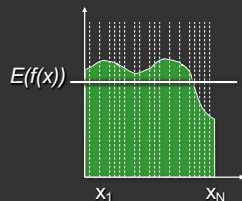
## Variance for Dice Example?

- Work out on board (variance for single dice roll)

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## Variance

$$\text{Var}[E(f(x))] = \frac{1}{N} \text{Var}[f(x)]$$

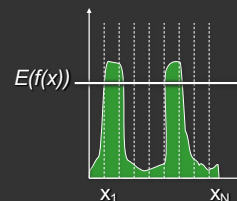


Variance decreases as 1/N  
Error decreases as 1/sqrt(N)

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## Variance

- Problem: variance decreases with 1/N
- Increasing # samples removes noise slowly



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## Variance Reduction

### Efficiency measure

$$Efficiency \propto \frac{1}{Variance \bullet Cost}$$

### Techniques

- Importance sampling
- Sampling patterns: stratified, ...

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## Variance Reduction Techniques

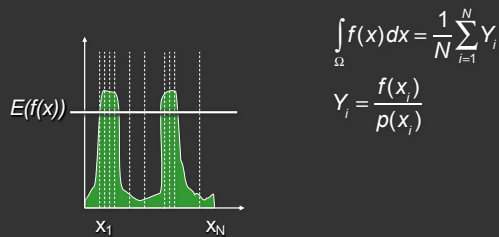
- Importance sampling
- Stratified sampling

$$\int_0^1 f(x) dx = \frac{1}{N} \sum_{i=1}^N f(x_i)$$

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## Importance Sampling

Put more samples where  $f(x)$  is bigger



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## Importance Sampling

- This is still unbiased

$$E[Y_i] = \int_{\Omega} Y(x) p(x) dx$$

$$= \int_{\Omega} \frac{f(x)}{p(x)} p(x) dx$$

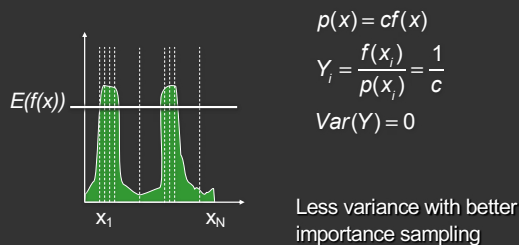
$$= \int_{\Omega} f(x) dx$$

for all N

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## Importance Sampling

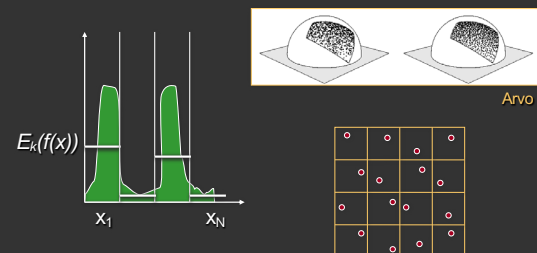
- Zero variance if  $p(x) \sim f(x)$



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## Stratified Sampling

- Estimate subdomains separately

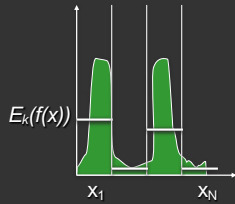


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## Stratified Sampling

- This is still unbiased

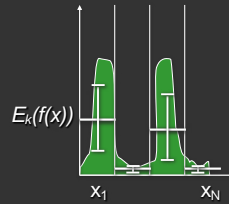


$$F_N = \frac{1}{N} \sum_{i=1}^N f(x_i) \\ = \frac{1}{N} \sum_{k=1}^M N_k F_i$$

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## Stratified Sampling

- Less overall variance if less variance in subdomains



$$Var[F_N] = \frac{1}{N^2} \sum_{i=1}^M N_i Var[F_i]$$

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## More Information

- Veach PhD thesis chapter (linked to from website)
- Course Notes (links from website)
  - *Mathematical Models for Computer Graphics*, Stanford, Fall 1997
  - *State of the Art in Monte Carlo Methods for Realistic Image Synthesis*, Course 29, SIGGRAPH 2001

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