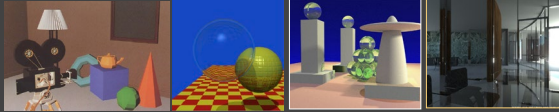


Computer Graphics II: Rendering

CSE 168 [Spr 21], Lecture 5: Monte Carlo Integration
Ravi Ramamoorthi

<http://viscomp.ucsd.edu/classes/cse168/sp21>



To Do

- Homework 2 (Direct Lighting) due Apr 22
- Assignment is on UCSD Online
- START EARLY (NOW)

Motivation

Rendering = integration

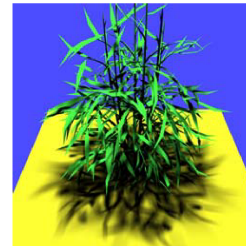
- Reflectance equation: Integrate over incident illumination
- Rendering equation: Integral equation

Many sophisticated shading effects involve integrals

- Antialiasing
- Soft shadows
- Indirect illumination
- Caustics

Example: Soft Shadows

$$E(x) = \int_{\mathbb{H}^2} L_i(x, \omega) \cos \theta d\omega$$



Challenges

- Visibility and blockers
- Varying light distribution
- Complex source geometry

Source: Agrawala, Ramamoorthi, Heirich, Moll, 2000

Monte Carlo

- Algorithms based on statistical sampling and random numbers
- Coined in the beginning of 1940s. Originally used for neutron transport, nuclear simulations
 - Von Neumann, Ulam, Metropolis, ...
- Canonical example: 1D integral done numerically
 - Choose a set of random points to evaluate function, and then average (expectation or statistical average)

Monte Carlo Algorithms

Advantages

- Robust for complex integrals in computer graphics (irregular domains, shadow discontinuities and so on)
- Efficient for high dimensional integrals (common in graphics: time, light source directions, and so on)
- Quite simple to implement
- Work for general scenes, surfaces
- Easy to reason about (but care taken re statistical bias)

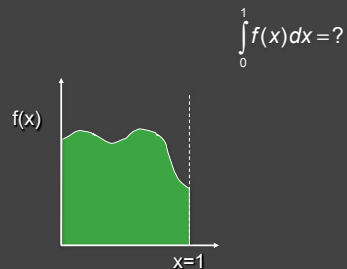
Disadvantages

- Noisy
- Slow (many samples needed for convergence)
- Not used if alternative analytic approaches exist (but those are rare)

Outline

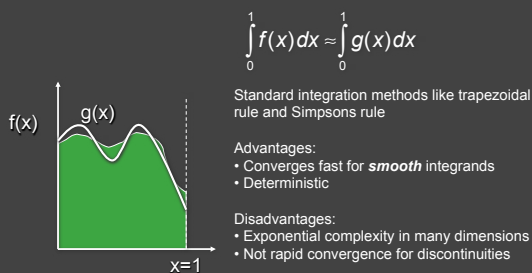
- Motivation
- Overview, 1D integration
- Basic probability and sampling
- Monte Carlo estimation of integrals

Integration in 1D



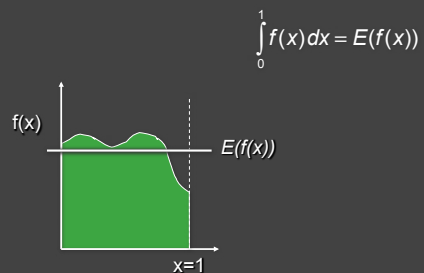
Slide courtesy of
Peter Shirley

We can approximate



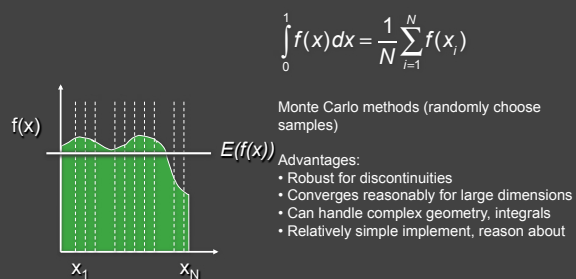
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Or we can average



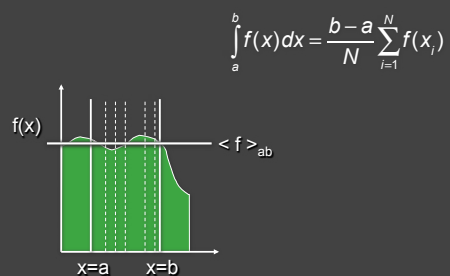
Slide courtesy of
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Estimating the average



Slide courtesy of
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Other Domains

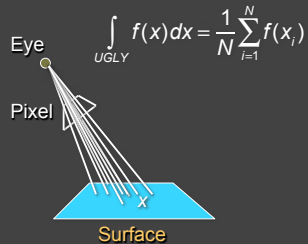


Slide courtesy of
Peter Shirley

Multidimensional Domains

Same ideas apply for integration over ...

- Pixel areas
- Surfaces
- Projected areas
- Directions
- Camera apertures
- Time
- Paths



Outline

- Motivation
- Overview, 1D integration
- *Basic probability and sampling*
- Monte Carlo estimation of integrals

Random Variables

- Describes possible outcomes of an experiment
- In discrete case, e.g. value of a dice roll [$x = 1-6$]
- Probability p associated with each x ($1/6$ for dice)
- Continuous case is obvious extension

Expected Value

- Expectation Discrete: $E(x) = \sum_{i=1}^n p_i x_i$
Continuous: $E(x) = \int_0^1 p(x) f(x) dx$

- For Dice example:

$$E(x) = \sum_{i=1}^n \frac{1}{6} x_i = \frac{1}{6} (1 + 2 + 3 + 4 + 5 + 6) = 3.5$$

Continuous Probability Distributions

PDF $p(x)$

$$p(x) \geq 0$$

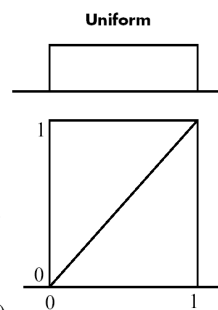
CDF $P(x)$

$$P(x) = \int_0^x p(x) dx$$

$$P(x) = \Pr(X < x) \quad P(1) = 1$$

$$\Pr(\alpha \leq X \leq \beta) = \int_{\alpha}^{\beta} p(x) dx$$

$$= P(\beta) - P(\alpha)$$



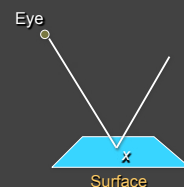
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Sampling Techniques

Problem: how do we generate random points/directions during path tracing?

- Non-rectilinear domains
- Importance (BRDF)
- Stratified



Generating Random Points

Uniform distribution:

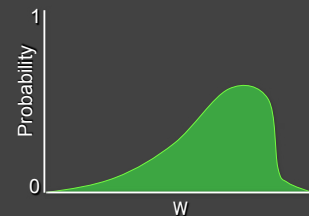
- Use random number generator



Generating Random Points

Specific probability distribution:

- Function inversion
- Rejection
- Metropolis



Common Operations

Want to **sample** probability distributions

- Draw samples distributed according to probability
- Useful for integration, picking important regions, etc.

Common distributions

- Disk or circle
- Uniform
- Upper hemisphere for visibility
- Area luminaire
- Complex lighting like an environment map
- Complex reflectance like a BRDF

Sampling Continuous Distributions

Cumulative probability distribution function

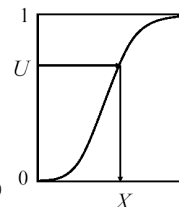
$$P(x) = \Pr(X < x)$$

Construction of samples

$$\text{Solve for } X = P^{-1}(U)$$

Must know:

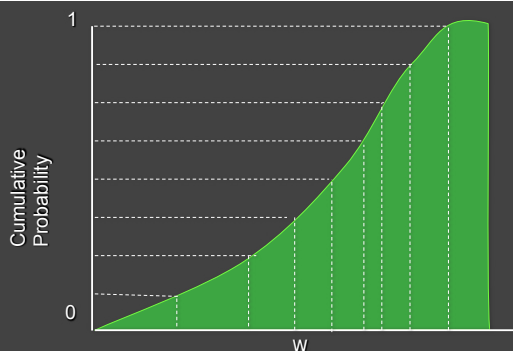
- The integral of $p(x)$
- The inverse function $P^{-1}(x)$



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Generating Random Points



Example: Power Function

Assume

$$p(x) = (n+1)x^n$$

$$P(x) = x^{n+1}$$

$$X \sim p(x) \Rightarrow X = P^{-1}(U) = \sqrt[n+1]{U}$$

$$\int_0^1 x^n dx = \frac{x^{n+1}}{n+1} \Big|_0^1 = \frac{1}{n+1}$$

Trick

$$Y = \max(U_1, U_2, \dots, U_n, U_{n+1})$$

$$\Pr(Y < x) = \prod_{i=1}^{n+1} \Pr(U_i < x) = x^{n+1}$$

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Sampling a Circle

$$A = \int_0^{2\pi} \int_0^1 r \, dr \, d\theta = \int_0^1 r \, dr \int_0^{2\pi} d\theta = \left(\frac{r^2}{2} \right) \Big|_0^1 \theta \Big|_0^{2\pi} = \pi$$

$$p(r, \theta) \, dr \, d\theta = \frac{1}{\pi} r \, dr \, d\theta \Rightarrow p(r, \theta) = \frac{r}{\pi}$$

$$p(r, \theta) = p(r)p(\theta)$$

$$p(\theta) = \frac{1}{2\pi}$$

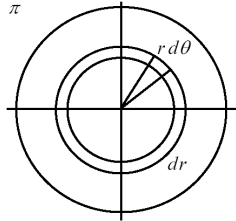
$$P(\theta) = \frac{1}{2\pi} \theta$$

$$p(r) = 2r$$

$$P(r) = r^2$$

$$\theta = 2\pi U_1$$

$$r = \sqrt{U_2}$$



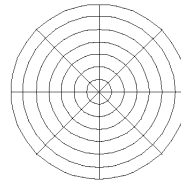
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Sampling a Circle

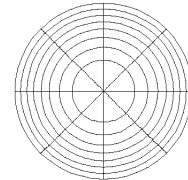
WRONG $\neq$ Equi-Areal

RIGHT = Equi-Areal



$$\theta = 2\pi U_1$$

$$r = U_2$$



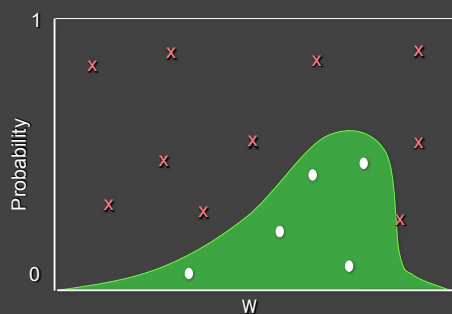
$$\theta = 2\pi U_1$$

$$r = \sqrt{U_2}$$

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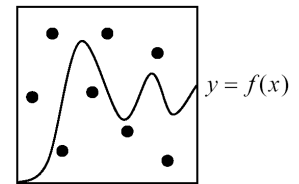
Rejection Sampling



Rejection Methods

$$I = \int_0^1 f(x) \, dx$$

$$= \iint_{y < f(x)} dx \, dy$$



Algorithm

Pick U_1 and U_2

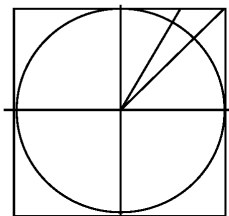
Accept U_1 if $U_2 < f(U_1)$

Wasteful? Efficiency = Area / Area of rectangle

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Sampling a Circle: Rejection



do {

$$X = 1 - 2 * U_1$$

$$Y = 1 - 2 * U_2$$

while ($X^2 + Y^2 > 1$)

May be used to pick random 2D directions

Circle techniques may also be applied to the sphere

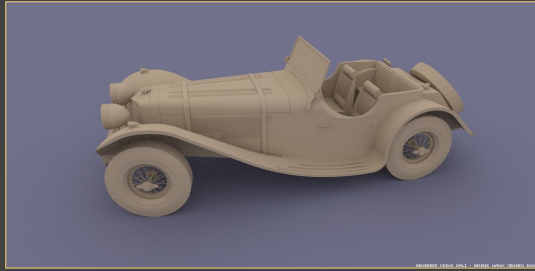
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Outline

- Motivation
- Overview, 1D integration
- Basic probability and sampling
- Monte Carlo estimation of integrals

Monte Carlo Path Tracing

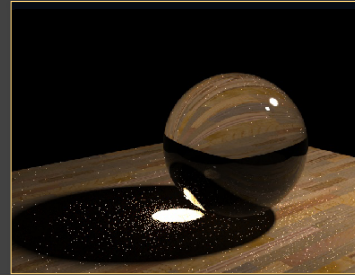


Big diffuse light source, 20 minutes

Motivation for rendering in graphics: Covered in detail in next lecture

Jensen

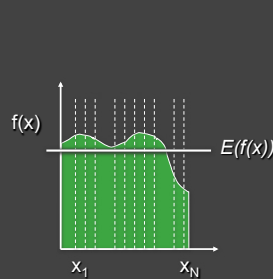
Monte Carlo Path Tracing



1000 paths/pixel

Jensen

Estimating the average



$$\int_0^1 f(x) dx = \frac{1}{N} \sum_{i=1}^N f(x_i)$$

Monte Carlo methods (randomly choose samples)

Advantages:

- Robust for discontinuities
- Converges reasonably for large dimensions
- Can handle complex geometry, integrals
- Relatively simple implement, reason about

Slide courtesy of Peter Shirley

Monte Carlo Integration

Definite integral	$I(f) \equiv \int_0^1 f(x) dx$
Expectation of f	$E[f] \equiv \int_0^1 f(x) p(x) dx$
Random variables	$X_i \sim p(x)$ $Y_i = f(X_i)$
Estimator	$F_N = \frac{1}{N} \sum_{i=1}^N Y_i$

Unbiased Estimator

$$E[F_N] = I(f)$$

Properties

$$E\left[\sum_i Y_i\right] = \sum_i E[Y_i]$$

$$E[aY] = aE[Y]$$

$$\begin{aligned} E[F_N] &= E\left[\frac{1}{N} \sum_{i=1}^N Y_i\right] \\ &= \frac{1}{N} \sum_{i=1}^N E[Y_i] = \frac{1}{N} \sum_{i=1}^N E[f(X_i)] \\ &= \frac{1}{N} \sum_{i=1}^N \int_0^1 f(x) p(x) dx \\ &= \frac{1}{N} \sum_{i=1}^N \int_0^1 f(x) dx \\ &= \int_0^1 f(x) dx \end{aligned}$$

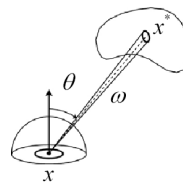
Assume uniform probability distribution for now

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Direct Lighting - Directional Sampling

$$E(x) = \int_{\Omega} L(x, \omega) \cos \theta d\omega$$



Ray intersection $x^*(x, \omega)$

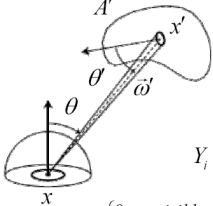
Sample ω uniformly by Ω

$$Y_i = L(x^*(x, \omega_i), -\omega_i) \cos \theta \cdot 2\pi$$

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Direct Lighting - Area Sampling

$$E(x) = \int_{\Omega} L_i(x, \omega) \cos \theta d\omega = \int_{A'} L_o(x', \omega') V(x, x') \frac{\cos \theta \cos \theta'}{|x - x'|^2} dA'$$


Ray direction $\omega' = x - x'$

Sample x' uniformly by A'

$$Y_i = L_o(x', \omega'_i) V(x, x'_i) \frac{\cos \theta \cos \theta'_i}{|x - x'_i|^2} A$$

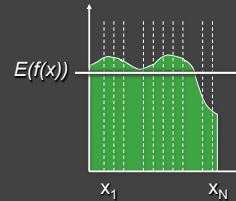
$$V(x, x') = \begin{cases} 0 & \text{--visible} \\ 1 & \text{visible} \end{cases}$$

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Variance

$$\text{Var}[f(x)] = \frac{1}{N} \sum_{i=1}^N [f(x_i) - E(f(x))]^2$$



Variance

Definition

$$\begin{aligned} V[Y] &\equiv E[(Y - E[Y])^2] \\ &= E[Y^2 - 2YE[Y] + E[Y]^2] \\ &= E[Y^2] - E[Y]^2 \end{aligned}$$

Properties

$$\begin{aligned} V[\sum_i Y_i] &= \sum_i V[Y_i] \\ V[aY] &= a^2 V[Y] \end{aligned}$$

Variance decreases with sample size

$$V\left[\frac{1}{N} \sum_{i=1}^N Y_i\right] = \frac{1}{N^2} \sum_{i=1}^N V[Y_i] = \frac{1}{N} V[Y]$$

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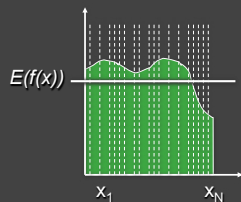
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Variance for Dice Example?

- Work out on board (variance for single dice roll)

Variance

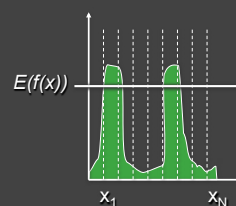
$$\text{Var}[E(f(x))] = \frac{1}{N} \text{Var}[f(x)]$$



Variance decreases as 1/N
Error decreases as 1/sqrt(N)

Variance

- Problem: variance decreases with 1/N
 - Increasing # samples removes noise slowly



Variance Reduction

Efficiency measure

$$Efficiency \propto \frac{1}{Variance \bullet Cost}$$

Techniques

- Importance sampling
- Sampling patterns: stratified, ...

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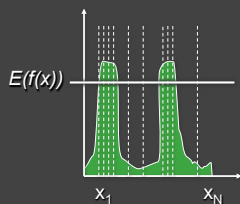
Variance Reduction Techniques

- Importance sampling
- Stratified sampling

$$\int_0^1 f(x) dx = \frac{1}{N} \sum_{i=1}^N f(x_i)$$

Importance Sampling

Put more samples where $f(x)$ is bigger

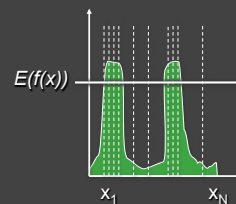


$$\int_{\Omega} f(x) dx = \frac{1}{N} \sum_{i=1}^N Y_i$$

$$Y_i = \frac{f(x_i)}{p(x_i)}$$

Importance Sampling

- This is still unbiased



$$E[Y_i] = \int_{\Omega} Y(x) p(x) dx$$

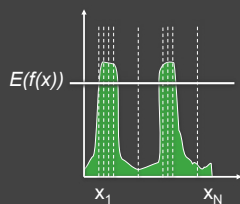
$$= \int_{\Omega} \frac{f(x)}{p(x)} p(x) dx$$

$$= \int_{\Omega} f(x) dx$$

for all N

Importance Sampling

- Zero variance if $p(x) \sim f(x)$



$$p(x) = cf(x)$$

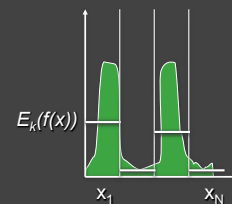
$$Y_i = \frac{f(x_i)}{p(x_i)} = \frac{1}{c}$$

$$Var(Y) = 0$$

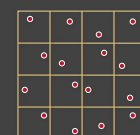
Less variance with better importance sampling

Stratified Sampling

- Estimate subdomains separately

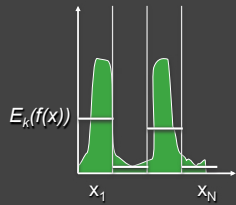


Arvo



Stratified Sampling

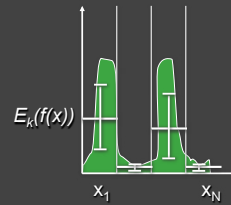
- This is still unbiased



$$F_N = \frac{1}{N} \sum_{i=1}^N f(x_i)$$
$$= \frac{1}{N} \sum_{k=1}^M N_k F_k$$

Stratified Sampling

- Less overall variance if less variance in subdomains



$$Var[F_N] = \frac{1}{N^2} \sum_{i=1}^M N_i Var[F_i]$$

More Information

- Veach PhD thesis chapter (linked to from website)
- Course Notes (links from website)
 - *Mathematical Models for Computer Graphics*, Stanford, Fall 1997
 - *State of the Art in Monte Carlo Methods for Realistic Image Synthesis*, Course 29, SIGGRAPH 2001