

Computer Graphics

CSE 167 [Win 24], Lecture 2: Review of Basic Math
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<http://viscomp.ucsd.edu/classes/cse167/wi24>

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To Do

- Complete Assignment 0 (due Jan 17)
- Get help if issues with compiling, programming
 - Remember to test HW 1,2,3 (and compile scratch for 4)
- Any problems with UCSD Online?
- Any confusion on course requirements?
- Textbooks: access to OpenGL references
- About first few lectures
 - Somewhat technical: core math ideas in graphics
 - HW1 is simple (only few lines of code): Lets you see how to use some ideas discussed in lecture, create images

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Motivation and Outline

- Many graphics concepts need basic math like linear algebra
 - Vectors (dot products, cross products, ...)
 - Matrices (matrix-matrix, matrix-vector mult., ...)
 - E.g: a point is a vector, and an operation like translating or rotating points on object can be matrix-vector multiply
- Should be refresher on very basic material for most of you
 - Only basic high school math required
 - If you don't understand, talk to me (review in office hours)

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Vectors

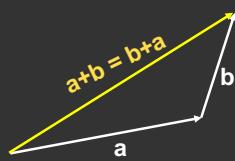


Usually written as \vec{a} or in bold. Magnitude written as $\|\vec{a}\|$

- Length and direction. Absolute position not important
- Use to store offsets, displacements, locations
 - But strictly speaking, positions are not vectors and cannot be added: a location implicitly involves an origin, while an offset does not.

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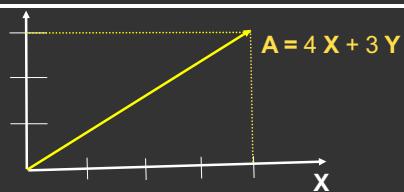
Vector Addition



- Geometrically: Parallelogram rule
- In cartesian coordinates (next), simply add coords

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Cartesian Coordinates



- X and Y can be any (usually orthogonal **unit**) vectors

$$A = \begin{pmatrix} x \\ y \end{pmatrix} \quad A^T = \begin{pmatrix} x & y \end{pmatrix} \quad \|A\| = \sqrt{x^2 + y^2}$$

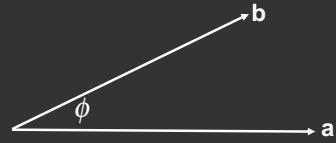
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Vector Multiplication

- Dot product
- Cross product
- Orthonormal bases and coordinate frames
- Note: Some books talk about right and left-handed coordinate systems. We *always* use right-handed

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Dot (scalar) product



$$\begin{aligned} \mathbf{a} \cdot \mathbf{b} &= \mathbf{b} \cdot \mathbf{a} = ? & \mathbf{a} \cdot \mathbf{b} &= \|\mathbf{a}\| \|\mathbf{b}\| \cos \phi \\ \mathbf{a} \cdot (\mathbf{b} + \mathbf{c}) &= \mathbf{a} \cdot \mathbf{b} + \mathbf{a} \cdot \mathbf{c} & \phi &= \cos^{-1} \left(\frac{\mathbf{a} \cdot \mathbf{b}}{\|\mathbf{a}\| \|\mathbf{b}\|} \right) \\ (k\mathbf{a}) \cdot \mathbf{b} &= k(\mathbf{a} \cdot \mathbf{b}) = k(\mathbf{a} \cdot \mathbf{b}) \end{aligned}$$

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Dot product in Cartesian components

$$\mathbf{a} \cdot \mathbf{b} = \begin{pmatrix} x_a \\ y_a \end{pmatrix} \bullet \begin{pmatrix} x_b \\ y_b \end{pmatrix} = ?$$

$$\mathbf{a} \cdot \mathbf{b} = \begin{pmatrix} x_a \\ y_a \end{pmatrix} \bullet \begin{pmatrix} x_b \\ y_b \end{pmatrix} = x_a x_b + y_a y_b$$

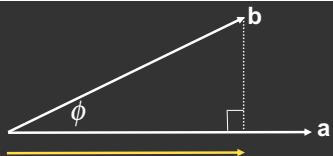
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Dot product: some applications in CG

- Find angle between two vectors (e.g. cosine of angle between light source and surface for shading)
- Finding projection of one vector on another (e.g. coordinates of point in arbitrary coordinate system)
- Advantage: computed easily in cartesian components

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Projections (of b on a)



$$\begin{aligned} \|\mathbf{b} \rightarrow \mathbf{a}\| &= ? & \|\mathbf{b} \rightarrow \mathbf{a}\| &= \|\mathbf{b}\| \cos \phi = \frac{\mathbf{a} \cdot \mathbf{b}}{\|\mathbf{a}\|} \\ \mathbf{b} \rightarrow \mathbf{a} &= ? & \mathbf{b} \rightarrow \mathbf{a} &= \|\mathbf{b} \rightarrow \mathbf{a}\| \frac{\mathbf{a}}{\|\mathbf{a}\|} = \frac{\mathbf{a} \cdot \mathbf{b}}{\|\mathbf{a}\|^2} \mathbf{a} \end{aligned}$$

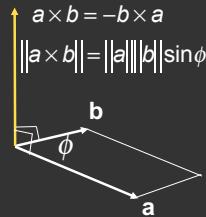
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Vector Multiplication

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Cross (vector) product



- Cross product orthogonal to two initial vectors
- Direction determined by right-hand rule
- Useful in constructing coordinate systems (later)

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Cross product: Properties

$$\begin{array}{ll}
 \mathbf{x} \times \mathbf{y} = +\mathbf{z} & \mathbf{a} \times \mathbf{b} = -\mathbf{b} \times \mathbf{a} \\
 \mathbf{y} \times \mathbf{x} = -\mathbf{z} & \mathbf{a} \times \mathbf{a} = 0 \\
 \mathbf{y} \times \mathbf{z} = +\mathbf{x} & \mathbf{a} \times (\mathbf{b} + \mathbf{c}) = \mathbf{a} \times \mathbf{b} + \mathbf{a} \times \mathbf{c} \\
 \mathbf{z} \times \mathbf{y} = -\mathbf{x} & \mathbf{a} \times (k\mathbf{b}) = k(\mathbf{a} \times \mathbf{b}) \\
 \mathbf{z} \times \mathbf{x} = +\mathbf{y} & \mathbf{x} \times \mathbf{z} = -\mathbf{y} \\
 \end{array}$$

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Cross product: Cartesian formula?

$$\mathbf{a} \times \mathbf{b} = \begin{vmatrix} \mathbf{x} & \mathbf{y} & \mathbf{z} \\ \mathbf{x}_a & \mathbf{y}_a & \mathbf{z}_a \\ \mathbf{x}_b & \mathbf{y}_b & \mathbf{z}_b \end{vmatrix} = \begin{pmatrix} \mathbf{y}_a \mathbf{z}_b - \mathbf{y}_b \mathbf{z}_a \\ \mathbf{z}_a \mathbf{x}_b - \mathbf{x}_a \mathbf{z}_b \\ \mathbf{x}_a \mathbf{y}_b - \mathbf{y}_a \mathbf{x}_b \end{pmatrix}$$

$$\mathbf{a} \times \mathbf{b} = \mathbf{A}^* \mathbf{b} = \begin{pmatrix} 0 & -\mathbf{z}_a & \mathbf{y}_a \\ \mathbf{z}_a & 0 & -\mathbf{x}_a \\ -\mathbf{y}_a & \mathbf{x}_a & 0 \end{pmatrix} \begin{pmatrix} \mathbf{x}_b \\ \mathbf{y}_b \\ \mathbf{z}_b \end{pmatrix}$$

Dual matrix of vector a

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Vector Multiplication

- Dot product
- Cross product
- *Orthonormal bases and coordinate frames*

▪ Note: book talks about right and left-handed coordinate systems. We *always* use right-handed

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Orthonormal bases/coordinate frames

- Important for representing points, positions, locations
- Often, many sets of coordinate systems (not just X, Y, Z)
 - Global, local, world, model, parts of model (head, hands, ...)
- Critical issue is transforming between these systems/bases
 - Topic of next 3 lectures

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Coordinate Frames

- Any set of 3 vectors (in 3D) so that

$$\begin{aligned}
 \|\mathbf{u}\| &= \|\mathbf{v}\| = \|\mathbf{w}\| = 1 \\
 \mathbf{u} \cdot \mathbf{v} &= \mathbf{v} \cdot \mathbf{w} = \mathbf{u} \cdot \mathbf{w} = 0 \\
 \mathbf{w} &= \mathbf{u} \times \mathbf{v}
 \end{aligned}$$

$$\mathbf{p} = (\mathbf{p} \cdot \mathbf{u})\mathbf{u} + (\mathbf{p} \cdot \mathbf{v})\mathbf{v} + (\mathbf{p} \cdot \mathbf{w})\mathbf{w}$$

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Constructing a coordinate frame

- Often, given a vector \mathbf{a} (viewing direction in HW1), want to construct an orthonormal basis
- Need a second vector \mathbf{b} (up direction of camera in HW1)
- Construct an orthonormal basis (for instance, camera coordinate frame to transform world objects into in HW1)

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Constructing a coordinate frame?

We want to associate \mathbf{w} with \mathbf{a} , and \mathbf{v} with \mathbf{b}

- But \mathbf{a} and \mathbf{b} are neither orthogonal nor unit norm
- And we also need to find \mathbf{u}

$$\mathbf{w} = \frac{\mathbf{a}}{\|\mathbf{a}\|}$$

$$\mathbf{u} = \frac{\mathbf{b} \times \mathbf{w}}{\|\mathbf{b} \times \mathbf{w}\|}$$

$$\mathbf{v} = \mathbf{w} \times \mathbf{u}$$

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Matrices

- Can be used to transform points (vectors)
 - Translation, rotation, shear, scale(more detail next lecture)

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What is a matrix

- Array of numbers ($m \times n = m$ rows, n columns)

$$\begin{pmatrix} 1 & 3 \\ 5 & 2 \\ 0 & 4 \end{pmatrix}$$

- Addition, multiplication by a scalar simple: element by element

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Matrix-matrix multiplication

- Number of columns in first must = rows in second

$$\begin{pmatrix} 1 & 3 \\ 5 & 2 \\ 0 & 4 \end{pmatrix} \begin{pmatrix} 3 & 6 & 9 & 4 \\ 2 & 7 & 8 & 3 \end{pmatrix}$$

- Element (i,j) in product is dot product of row i of first matrix and column j of second matrix

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Matrix-matrix multiplication

- Number of columns in first must = rows in second

$$\begin{pmatrix} 1 & 3 \\ 5 & 2 \\ 0 & 4 \end{pmatrix} \begin{pmatrix} 3 & 6 & 9 & 4 \\ 2 & 7 & 8 & 3 \end{pmatrix} = \begin{pmatrix} 9 & 27 & 33 & 13 \\ 19 & 44 & 61 & 26 \\ 8 & 28 & 32 & 12 \end{pmatrix}$$

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Matrix-matrix multiplication

- Number of columns in first must = rows in second

$$\begin{pmatrix} 3 & 6 & 9 & 4 \\ 2 & 7 & 8 & 3 \end{pmatrix} \begin{pmatrix} 1 & 3 \\ 5 & 2 \\ 0 & 4 \end{pmatrix} \text{ NOT EVEN LEGAL!!}$$

- Non-commutative (AB and BA are different in general)
- Associative and distributive
 - $A(B+C) = AB + AC$
 - $(A+B)C = AC + BC$

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Matrix-Vector Multiplication

- Key for transforming points (next lecture)
- Treat vector as a column matrix ($m \times 1$)
- E.g. 2D reflection about y-axis

$$\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -x \\ y \end{pmatrix}$$

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Transpose of a Matrix (or vector?)

$$\begin{pmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{pmatrix}^T = \begin{pmatrix} 1 & 3 & 5 \\ 2 & 4 & 6 \end{pmatrix}$$

$$(AB)^T = B^T A^T$$

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Identity Matrix and Inverses

$$I_{3 \times 3} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$AA^{-1} = A^{-1}A = I$$

$$(AB)^{-1} = B^{-1}A^{-1}$$

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Vector multiplication in Matrix form

- Dot product?

$$a \bullet b = a^T b$$

$$\begin{pmatrix} x_a & y_a & z_a \end{pmatrix} \begin{pmatrix} x_b \\ y_b \\ z_b \end{pmatrix} = (x_a x_b + y_a y_b + z_a z_b)$$

- Cross product?

$$a \times b = A^* b = \begin{pmatrix} 0 & -z_a & y_a \\ z_a & 0 & -x_a \\ -y_a & x_a & 0 \end{pmatrix} \begin{pmatrix} x_b \\ y_b \\ z_b \end{pmatrix}$$

Dual matrix of vector a